

## 2020 考研数学二真题及解析完整版

来源：文都教育

一、选择题：1~8 小题，第小题 4 分，共 32 分。下列每题给出四个选项中，只有一个选项是符合题目要求的，请将选项前的字母填在答题纸指定位置上。

1.  $x \rightarrow 0^+$ ，下列无穷小量中最高阶是（ ）

A.  $\int_0^x (e^{t^2} - 1) dt$

B.  $\int_0^x \ln(1 + \sqrt{t^3}) dt$

C.  $\int_0^{\sin x} \sin t^2 dt$

D.  $\int_0^{1-\cos x} \sqrt{\sin^3 t} dt$

答案：D

解析：A.  $\int_0^x (e^{t^2} - 1) dt \sim \int_0^x t^2 dt = \frac{1}{3}x^3$

B.  $\int_0^x \ln(1 + \sqrt{t^3}) dt \sim \int_0^x \frac{1}{2}t^{3/2} dt = \frac{2}{5}x^{5/2}$

C.  $\int_0^{\sin x} \sin t^2 dt \sim \int_0^x t^2 dt = \frac{1}{3}x^3$

D.  $\int_0^{1-\cos x} \sqrt{\sin^3 t} dt \sim \int_0^{\frac{1}{2}x^2} t^{3/2} dt = \frac{2}{5}t^{5/2} \Big|_0^{\frac{1}{2}x^2}$

$$= \frac{2}{5} \left( \frac{1}{2}x^2 \right)^{5/2} = \frac{1}{10\sqrt{2}}x^5$$

2.  $f(x) = \frac{1}{(e^x - 1)(x - 2)} \ln|1 + x|$  第二类间断点个数（ ）

A. 1

B. 2

C. 3

D. 4

答案：C

解析： $x = 0, x = 2, x = 1, x = -1$  为间断点

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \frac{e^{\frac{1}{x-1}} \ln |1+x|}{(e^x - 1)(x-2)} = \lim_{x \rightarrow 0} \frac{e^{-1} \ln |1+x|}{-2x} = -\frac{e^{-1}}{2} \lim_{x \rightarrow 0} \frac{\ln |x+1|}{x} = -\frac{e^{-1}}{2}$$

$x = 0$  为可去间断点

$$\lim_{x \rightarrow 2} f(x) = \lim_{x \rightarrow 2} \frac{e^{\frac{1}{x-1}} \ln |1+x|}{(e^x - 1)(x-2)} = \infty$$

$x = 2$  为第二类间断点

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} \frac{e^{\frac{1}{x-1}} \ln |1+x|}{(e^x - 1)(x-2)} = 0$$

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} \frac{e^{\frac{1}{x-1}} \ln |1+x|}{(e^x - 1)(x-2)} = \infty$$

$x = 1$  为第二类间断点

$$\lim_{x \rightarrow -1} f(x) = \lim_{x \rightarrow -1} \frac{e^{\frac{1}{x-1}} \ln |1+x|}{(e^x - 1)(x-2)} = \infty$$

$x = -1$  为第二类间断点

3.  $\int_0^1 \frac{\arcsin \sqrt{x}}{\sqrt{x(1-x)}} dx$

A.  $\frac{\pi}{4}$

B.  $\frac{\pi^2}{8}$

C.  $\frac{\pi}{4}$

D.  $\frac{\pi}{8}$

答案: A

解析:

$$\int_0^1 \frac{\arcsin \sqrt{x}}{\sqrt{x(1-x)}} dx$$

令  $u = \sqrt{x}$ , 则

$$\text{原式} = \int_0^1 \frac{\arcsin u}{\sqrt{u^2(1-u^2)}} \cdot 2u du$$

$$= 2 \int_0^1 \frac{\arcsin u}{\sqrt{1-u^2}} du$$

$$\text{令 } u = \sin t \Rightarrow \int_0^{\frac{\pi}{2}} \frac{t}{\cos t} \cos t dt$$

$$= 2 \cdot \frac{1}{2} t^2 \Big|_0^{\frac{\pi}{2}} = \frac{\pi^2}{4}$$

4.  $f(x) = x^2 \ln(1-x)$ ,  $n \geq 3$  时,  $f^{(n)}(0) =$

A.  $-\frac{n!}{n-2}$

B.  $\frac{n!}{n-2}$

C.  $-\frac{(n-2)!}{n}$

D.  $\frac{(n-2)!}{n}$

答案: A

解析:

$$f(x) = x^2 \ln(1-x), n \geq 3$$

$$f^{(n)}(x) = C_n^0 [x^2 \ln(1-x)]^{(n)} + C_n^1 (x^2)' [\ln(1-x)]^{(n-1)} + C_n^2 (x^2)'' [\ln(1-x)]^{(n-2)}$$

$$= \frac{(n-1)!(-1)}{(1-x)^n}$$

$$+ 2n \cdot x \cdot \frac{(n-2)!(-1)}{(1-x)^{n-1}}$$

$$+ 2 \cdot \frac{(n-3)!(-1)}{(1-x)^{n-2}}$$

$$(x^2)' = 2x; (x^2)'' = 2.$$

$$\therefore f^{(n)}(x) = x^2 \cdot \frac{(n-1)!(-1)}{(1-x)^n} + 2n \cdot x \cdot \frac{(n-2)!(-1)}{(1-x)^{n-1}} + 2 \cdot \frac{(n-3)!(-1)}{(1-x)^{n-2}}$$

$$\therefore f^{(n)}(0) = -\frac{n!}{n-2}$$

5. 关于函数  $f(x, y) = \begin{cases} xy, & xy \neq 0 \\ 0, & xy = 0 \end{cases}$  给出以下结论

①  $\left. \frac{\partial f}{\partial x} \right|_{(0,0)} = 1$

$$\textcircled{2} \left. \frac{\partial^2 f}{\partial x \partial y} \right|_{(0,0)} = 1$$

$$\textcircled{3} \lim_{(x,y) \rightarrow (0,0)} f(x,y) = 0$$

④  $\lim_{y \rightarrow 0} \lim_{x \rightarrow 0} f(x,y) = 0$  正确的个数是

- A.4
- B.3
- C.2
- D.1

答案：B

解析：

$$\textcircled{1} \left. \frac{\partial f}{\partial x} \right|_{(0,0)} = \lim_{x \rightarrow 0} \frac{f(x,0) - f(0,0)}{x} = \lim_{x \rightarrow 0} \frac{x-0}{x} = 1$$

$$\textcircled{2} xy \neq 0 \text{ 时, } \frac{\partial f}{\partial y} = y$$

$$y = 0 \text{ 时, } \frac{\partial f}{\partial y} = 0$$

$$\left. \frac{\partial^2 f}{\partial x \partial y} \right|_{(0,0)} = \lim_{y \rightarrow 0} \frac{f'_x(0,y) - f'_x(0,0)}{y} = \lim_{y \rightarrow 0} \frac{-1}{y} \text{ 不存在.}$$

$$\textcircled{3} xy \neq 0, \lim_{(x,y) \rightarrow (0,0)} f(x,y) = \lim_{(x,y) \rightarrow (0,0)} xy = 0$$

$$y = 0, \lim_{(x,y) \rightarrow (0,0)} f(x,y) = \lim_{(x,y) \rightarrow (0,0)} x = 0$$

$$x = 0, \lim_{(x,y) \rightarrow (0,0)} f(x,y) = \lim_{(x,y) \rightarrow (0,0)} y = 0$$

$$\therefore \lim_{(x,y) \rightarrow (0,0)} f(x,y) = 0$$

$$\textcircled{4} xy \neq 0, \lim_{x \rightarrow 0} f(x,y) = \lim_{x \rightarrow 0} y = y$$

$$y = 0, \lim_{x \rightarrow 0} f(x,y) = \lim_{x \rightarrow 0} x = 0$$

$$x = 0, \lim_{x \rightarrow 0} f(x,y) = \lim_{x \rightarrow 0} y = y$$

从而  $\lim_{y \rightarrow 0} \lim_{x \rightarrow 0} f(x,y) = 0$ .

6. 设函数  $f(x)$  在区间  $[-2, 2]$  上可导, 且  $f'(x) > f(x) > 0$ , 则 ( )

A.  $\frac{f(-2)}{f(-1)} > 1$

B.  $\frac{f(0)}{f(-1)} > e$

C.  $\frac{f(1)}{f(-1)} < e^2$

D.  $\frac{f(2)}{f(-1)} < e^3$

答案: B

解析: 由  $f'(x) > f(x) > 0$  知

$$\frac{f'(x)}{f(x)} - 1 > 0$$

即  $(\ln f(x) - x)' > 0$

令  $F(x) = \ln f(x) - x$ , 则  $F(x)$  在  $[-2, 2]$  上单增

因  $-2 < -1, \ln f(-2) < F(-1) < F(-2)$

即  $\ln f(-2) + 2 < \ln f(-1) + 1$

$$\frac{f(-1)}{f(-2)} > e$$

同理  $\ln f(-1) < F(0)$

即  $\ln f(-1) + 1 < \ln f(0)$

$$\frac{f(0)}{f(-1)} > e$$

7. 设四阶矩阵  $A = (a_{ij})$  不可逆,  $a_{12}$  的代数余子式  $A_{12} = 0, \alpha_1, \alpha_2, \alpha_3, \alpha_4$  为矩阵  $A$  的列向量

组,  $A^*$  为  $A$  的伴随矩阵, 则方程组  $A^*x = 0$  的通解为 ( )

A.  $x = k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3$ , 其中  $k_1, k_2, k_3$  为任意常数

B.  $x = k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_4$ , 其中  $k_1, k_2$  为任意常数

C.  $x = k_1\alpha_1 + k_2\alpha_3 + k_3\alpha_4$ , 其中  $k_1, k_2, k_3$  为任意常数

D.  $x = k_1\alpha_2 + k_2\alpha_3 + k_3\alpha_4$ , 其中  $k_1, k_2, k_3$  为任意常数

答案: C

解析:

$\because A$  不可逆

$\therefore |A|=0$

$\therefore A_{12} \neq 0 \quad \therefore r(A)=3$

$\therefore r(A^*)=1$

$\therefore A^*x=0$  的基础解系有 3 个线性无关的解向量.

$\therefore A^*A=|A|E=0$

$\therefore A$  的每一列都是  $A^*x=0$  的解

又  $\therefore A_{12} \neq 0 \quad \therefore \alpha_1, \alpha_3, \alpha_4$  线性无关

$\therefore A^*x=0$  的通解为  $x=k_1\alpha_1+k_2\alpha_3+k_3\alpha_4$

8. 设  $A$  为 3 阶矩阵,  $\alpha_1, \alpha_2$  为  $A$  属于特征值 1 的线性无关的特征向量,  $\alpha_3$  为  $A$  的属于特征

值 -1 的特征向量, 则满足  $P^{-1}AP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$  的可逆矩阵  $P$  可为 ( ).

A.  $(\alpha_1, \alpha_3, -\alpha_3)$

B.  $(\alpha_1, \alpha_2, -\alpha_3)$

C.  $(\alpha_1, -\alpha_2, \alpha_3)$

D.  $(\alpha_1 + \alpha_2, -\alpha_3, -\alpha_2)$

答案: D

解析:

$A\alpha_1 = \alpha_1, A\alpha_2 = \alpha_2$

$A\alpha_3 = -\alpha_3$

$\therefore P^{-1}AP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\therefore P$  的 1, 3 两列为 1 的线性无关的特征向量  $\alpha_1 + \alpha_2, \alpha_2$

$P$  的第 2 列为  $A$  的属于 -1 的特征向量  $\alpha_3$ .

$\therefore P = (\alpha_1 + \alpha_2, -\alpha_3, \alpha_2)$

二、填空题：9~14 小题，每小题 4 分，共 24 分. 请将答案写在答题纸指定位置上.

9. 设  $\begin{cases} x = \sqrt{t^2 + 1} \\ y = \ln(t + \sqrt{t^2 + 1}) \end{cases}$ , 则  $\frac{d^2y}{dx^2} \Big|_{t=1} = \underline{\hspace{2cm}}$ .

解析:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{1}{t + \sqrt{t^2 + 1}} \left( 1 + \frac{t}{\sqrt{t^2 + 1}} \right)}{\frac{t}{\sqrt{t^2 + 1}}}$$

$$= \frac{1}{t}$$

$$\frac{d^2y}{dx^2} = \frac{d\left(\frac{dy}{dx}\right)}{dx} = \frac{d\left(\frac{1}{t}\right)}{\frac{dx}{dt}} = \frac{-\frac{1}{t^2}}{\frac{t}{\sqrt{t^2 + 1}}}$$

$$= -\frac{\sqrt{t^2 + 1}}{t^3}$$

$$\frac{d^2y}{dx^2} \Big|_{t=1} = -\frac{\sqrt{1^2 + 1}}{1^3} = -\sqrt{2}$$

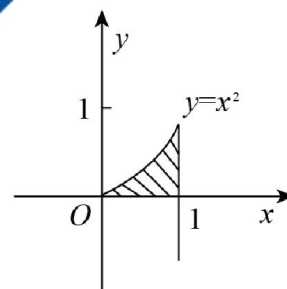
10.  $\int_0^1 dy \int_{\sqrt{y}}^1 \sqrt{x^3 + 1} dx = \underline{\hspace{2cm}}$ .

解析:  $\int_0^1 dy \int_{\sqrt{y}}^1 \sqrt{x^3 + 1} dx$

$$= \int_0^1 dx \int_0^{x^2} \sqrt{x^3 + 1} dy$$

$$= \int_0^1 \sqrt{x^3 + 1} dx \int_0^{x^2} dy$$

$$= \int_0^1 \sqrt{x^3 + 1} x^2 dx$$



$$\begin{aligned}
 &= \frac{1}{3} \int_0^1 (x^3 + 1)^{\frac{1}{2}} d(x^3 + 1) \\
 &= \frac{1}{3} \cdot \frac{2}{3} (x^3 + 1)^{\frac{3}{2}} \Big|_0^1 \\
 &= \frac{2}{9} \left( 2^{\frac{3}{2}} - 1 \right)
 \end{aligned}$$

11. 设  $z = \arctan[xy + \sin(x + y)]$ , 则  $dz|_{(0,\pi)} =$  \_\_\_\_\_

解析:

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

$$\frac{\partial z}{\partial x} = \frac{1}{1 + [xy + \sin(x + y)]^2} [y + \cos(x + y)] \Big|_{(0,\pi)} = \pi - 1$$

$$\frac{\partial z}{\partial y} = \frac{1}{1 + [xy + \sin(x + y)]^2} [x + \cos(x + y)] \Big|_{(0,\pi)} = -1$$

$$\therefore \frac{\partial z}{\partial x} dx - \frac{\partial z}{\partial y} dy = (\pi - 1) dx - dy$$

12. 斜边长为  $a$  等腰直角三角形平板铅直地沉没在水中, 且斜边与水面相齐, 平板重量为  $G$ , 速度为  $v$ , 水的密度为  $\rho$ , 该平板一侧所受的水压力为 \_\_\_\_\_

解: 建立直角坐标系, 如图所示

$$\begin{aligned}
 F &= 2 \int_0^a \rho g \cdot x \cdot (a - x) dx \\
 &= 2 \rho g \int_0^a (ax - x^2) dx \\
 &= 2 \rho g \left( \frac{a}{2} x^2 - \frac{1}{3} x^3 \right) \Big|_0^a \\
 &= \frac{1}{3} \rho g a^3
 \end{aligned}$$

13. 设  $y = y(x)$  满足  $y'' + 2y' + y = 0$ ,  $y(0) = 0, y'(0) = 1$ , 则  $\int_0^{+\infty} y(x) dx =$  \_\_\_\_\_

解析: 特征方程  $\lambda^2 + 2\lambda + 1 = 0$

$$\therefore \lambda_1 = \lambda_2 = -1$$

$$\therefore y(x) = (C_1 + C_2 x) e^{-x}$$



$$\int_0^{+\infty} y(x) dx = -\int_0^{+\infty} y''(x) + 2y'(x) dx$$

$$= -[y'(x) + 2y(x)] \Big|_0^{+\infty}$$

$$= [y'(0) + 2y(0)] = 1$$

14. 行列式  $\begin{vmatrix} a & 0 & -1 & 1 \\ 0 & a & 1 & -1 \\ -1 & 1 & a & 0 \\ 1 & -1 & 0 & a \end{vmatrix} = \underline{\hspace{2cm}}$

解析:

$$\begin{vmatrix} a & 0 & -1 & 1 \\ 0 & a & 1 & -1 \\ -1 & 1 & a & 0 \\ 1 & -1 & 0 & a \end{vmatrix} = \begin{vmatrix} a & 0 & -1 & 1 \\ 0 & a & 1 & -1 \\ -1 & 1 & a & 0 \\ 0 & 0 & a & a \end{vmatrix}$$

$$= \begin{vmatrix} 0 & a & -1+a^2 & -1+a \\ 0 & a & 1 & -1 \\ -1 & 1 & a & 0 \\ 0 & 0 & a & a \end{vmatrix}$$

$$= \begin{vmatrix} a & a^2 & -1 & -1 \\ -a & -a^2 & -1 & -1 \\ -1 & 1 & a & 0 \\ 0 & 0 & a & a \end{vmatrix} = a^4 - 4a^2.$$

三、解答题：15~20 小题，共 94 分。请将解答写在答题纸指定位置。解答应写出文字说明、证明过程或演算步骤。

15. (本题满分 10 分)

求曲线  $y = \frac{x^{1+x}}{(1+x)^x}$  ( $x > 0$ ) 的斜渐近线方程。

解析:  $\lim_{x \rightarrow +\infty} \frac{y}{x} = \lim_{x \rightarrow +\infty} \frac{x^{1+x}}{(1+x)^x x}$

$$= \lim_{x \rightarrow +\infty} \frac{x^x}{(1+x)^x}$$

$$= \lim_{x \rightarrow +\infty} \frac{e^{x \ln x}}{e^{x \ln(1+x)}}$$

$$= \lim_{x \rightarrow +\infty} e^{x(\ln x - \ln(1+x))}$$

$$= \lim_{x \rightarrow +\infty} e^{x \ln \frac{x+1}{1+x}}$$

$$= \lim_{x \rightarrow +\infty} e^{x \ln \left(1 - \frac{1}{1+x}\right)}$$

$$= \lim_{x \rightarrow +\infty} e^{x \left(-\frac{1}{1+x}\right)} = e^{-1}$$

$$\lim_{x \rightarrow +\infty} (y - e^{-1}x)$$

$$= \lim_{x \rightarrow +\infty} \left( \frac{x^{1+x}}{(1+x)^x} - e^{-1}x \right)$$

$$= \lim_{x \rightarrow +\infty} x \left( \frac{x^x}{(1+x)^x} - e^{-1} \right)$$

$$= \lim_{x \rightarrow +\infty} x \cdot \left( e^{\frac{x \ln x}{1+x}} - e^{-1} \right)$$

$$= \lim_{x \rightarrow +\infty} x \cdot \left( e^{\frac{x \ln x}{1+x}} - 1 \right)$$

$$= \lim_{x \rightarrow +\infty} x \ln \frac{x}{1+x} + 1$$

$$= \lim_{t \rightarrow 0^+} e^{-1} \frac{\frac{1}{t} \cdot \ln \frac{t}{1+t} + 1}{t}$$

$$= \lim_{t \rightarrow 0^+} e^{-1} \frac{\ln \frac{1}{t+1} + t}{t^2}$$

$$= \lim_{t \rightarrow 0^+} e^{-1} \frac{t - \ln(1+t)}{t^2} = \frac{1}{2} e^{-1}$$

∴ 曲线的斜渐近线方程为  $y = e^{-1}x + \frac{1}{2}e^{-1}$

16. (本题满分 10 分)

已知函数  $f(x)$  连续且  $\lim_{x \rightarrow 0} \frac{f(x)}{x} = 1$ ,  $g(x) = \int_0^1 f(xt) dt$ , 求  $g'(x)$  并证明  $g'(x)$  在  $x=0$  处连续.

解析：因为  $\lim_{x \rightarrow 0} \frac{f(x)}{x} = 1$   $\therefore f(0) = \lim_{x \rightarrow 0} f(x) = 0$

所以  $g(0) = \int_0^1 f(0) dt = 0$

因为  $g(x) = \int_0^1 f(xt) dt \stackrel{xt=u}{=} \frac{1}{x} \int_0^x f(u) du$

当  $x \neq 0$  时,  $g'(x) = \frac{xf(x) - \int_0^x f(u) du}{x^2}$

当  $x=0$  时,  $g'(0) = \lim_{x \rightarrow 0} \frac{g(x) - g(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{\int_0^x f(u) du}{x^2} = \lim_{x \rightarrow 0} \frac{\frac{1}{x} \int_0^x f(u) du}{x} = \lim_{x \rightarrow 0} \frac{1}{x} \lim_{x \rightarrow 0} \frac{\int_0^x f(u) du}{1} = \lim_{x \rightarrow 0} \frac{1}{x} \cdot 1 = \frac{1}{2}$

$$\therefore g'(x) = \begin{cases} \frac{\int_0^x f(u) du}{x^2}, & x \neq 0 \\ \frac{1}{2}, & x = 0 \end{cases}$$

又因为  $\lim_{x \rightarrow 0} g'(x) = \lim_{x \rightarrow 0} \frac{xf(x) - \int_0^x f(u) du}{x^2}$

$$= \lim_{x \rightarrow 0} \left[ \frac{f(x)}{x} - \frac{\int_0^x f(u) du}{x^2} \right] = 1 - \frac{1}{2} = \frac{1}{2}$$

$\therefore g'(x)$  在  $x=0$  处连续  
17 (本题满分 10 分)

求函数  $f(x, y) = x^3 + 8y^3 - xy$  的极值

解析：一阶导数可求

$$\frac{\partial f}{\partial x} = 3x^2 - y$$

$$\frac{\partial f}{\partial y} = 24y^2 - x$$

$$\text{令 } \begin{cases} \frac{\partial f}{\partial x} = 0 \\ \frac{\partial f}{\partial y} = 0 \end{cases} \text{ 可得 } \begin{cases} x = 0 \\ y = 0 \end{cases} \text{ 或 } \begin{cases} x = \frac{1}{6} \\ y = \frac{1}{12} \end{cases}$$

求二阶导可得

$$\frac{\partial^2 f}{\partial x^2} = 6x \quad \frac{\partial^2 f}{\partial x^2 y} = -1 \quad \frac{\partial^2 f}{\partial y^2} = 48y$$

当  $x=0, y=0$  时,  $A=0, B=-1, C=0$

$AC - B^2 < 0$  故不是极值.

当  $x = \frac{1}{6}, y = \frac{1}{12}$  时

$A = 1, B = -1, C = 4.$

$AC - B^2 > 0, A = 1 > 0$  故  $(\frac{1}{6}, \frac{1}{12})$  且极小值

极小值  $f(\frac{1}{6}, \frac{1}{12}) = (\frac{1}{6})^3 + 8(\frac{1}{12})^3 - 6 \times \frac{1}{12} = \frac{1}{216} + \frac{2}{27} - \frac{1}{2} = -\frac{1}{2}$

18. 已知  $2f(x) + x^2 f(\frac{1}{x}) = \frac{x^2 + 2x}{\sqrt{1+x^2}}$ , 求直线  $y = \frac{1}{2}, y = \frac{\sqrt{3}}{2}$  与函数  $f(x)$  所

围图形绕 x 轴旋转一周而成的旋转体的体积.

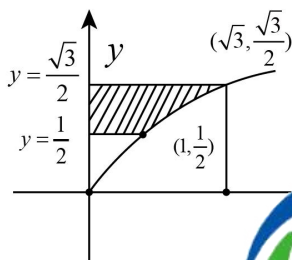
解析: ①  $\because 2f(x) + x^2 f(\frac{1}{x}) = \frac{x^2 + 2x}{\sqrt{1+x^2}}$

$$2f(\frac{1}{x}) + x^2 f(x) = \frac{\frac{1}{x^2} + 2 \cdot \frac{1}{x}}{\sqrt{1+\frac{1}{x^2}}} = \frac{1+2x}{x\sqrt{1+x^2}} \dots ②$$

② - ① 得

$$f(x) = \frac{x}{\sqrt{x^2+1}}$$

②

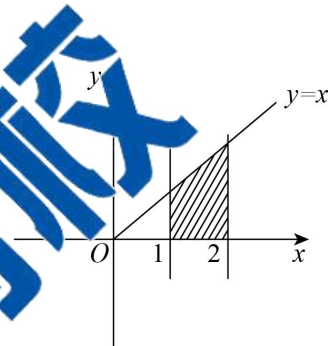


$$\begin{aligned} V &= \pi \cdot \left(\frac{\sqrt{3}}{2}\right)^2 \cdot \sqrt{3} - \pi \int_1^{\sqrt{3}} \frac{x}{x^2+1} dx \\ &= \frac{3\sqrt{3}}{4} \pi - \frac{1}{4} \pi - \pi \cdot \sqrt{3} + \frac{\pi}{12} \\ &= \frac{\pi^2}{12} - \frac{1}{4} \pi - \frac{\sqrt{3}}{4} \pi \end{aligned}$$

19. (本题满分 10 分)

平面 D 由直线  $x=1, x=2, y=x$  与  $x$  轴围成, 计算  $\iint_D \frac{\sqrt{x^2+y^2}}{x} dx dy$ .

解析: 积分区域如图:



$$\begin{aligned} & \iint_D \frac{\sqrt{x^2+y^2}}{x} dx dy \\ &= \int_0^{\frac{\pi}{4}} d\theta \int_{\frac{1}{\cos\theta}}^{\frac{2}{\cos\theta}} \frac{r}{r \cos\theta} \cdot r dr \\ &= \int_0^{\frac{\pi}{4}} \frac{1}{\cos\theta} \cdot \frac{1}{2} r^2 \Big|_{\frac{1}{\cos\theta}}^{\frac{2}{\cos\theta}} d\theta \\ &= \frac{1}{2} \int_0^{\frac{\pi}{4}} \frac{1}{\cos\theta} \cdot \frac{3}{\cos^2\theta} d\theta \\ &= \frac{3}{2} \int_0^{\frac{\pi}{4}} \sec^3\theta d\theta = \frac{3}{2} \int_0^{\frac{\pi}{4}} \sec\theta \tan\theta \\ &= \frac{3}{2} \left[ \theta \Big|_0^{\frac{\pi}{4}} - \frac{1}{2} \tan^2\theta \sec\theta \Big|_0^{\frac{\pi}{4}} \right] \\ &= \frac{3}{2} \left[ \frac{\pi}{4} - \frac{1}{2} (\sec^2\theta - 1) \sec\theta \Big|_0^{\frac{\pi}{4}} \right] \\ &= \frac{3}{2} \left( \sqrt{2} - \int_0^{\frac{\pi}{4}} \sec^3\theta d\theta + \int_0^{\frac{\pi}{4}} \sec\theta d\theta \right) \\ &= \frac{3}{2} \left( \sqrt{2} - \int_0^{\frac{\pi}{4}} \sec^3\theta d\theta + \ln|\sec\theta + \tan\theta| \Big|_0^{\frac{\pi}{4}} \right) \\ &= \frac{3}{2} \left( \sqrt{2} - \int_0^{\frac{\pi}{4}} \sec^3\theta d\theta + \ln(\sqrt{2}+1) \right) \\ & \text{所以 } \int_0^{\frac{\pi}{4}} \sec^3\theta d\theta = \frac{\sqrt{2}}{2} \ln(\sqrt{2}+1) \end{aligned}$$

$$\iint_D \frac{\sqrt{x^2+y^2}}{x} dx dy = \frac{3}{2} \left( \frac{\sqrt{2}}{2} + \frac{1}{2} \ln(\sqrt{2}+1) \right)$$

$$= \frac{3}{4} [\sqrt{2} + \ln(\sqrt{2} + 1)]$$

20. (本题满分 11 分)

设函数  $f(x) = \int_1^x e^{t^2} dt$ .

证: 存在  $\xi \in (1, 2), f(\xi) = (2 - \xi)e^{\xi^2}$ ;

(2) 证: 存在  $\eta \in (1, 2), f(2) = \ln 2 \cdot \eta e^{\eta^2}$ .

证明 (1) 构造辅助函数  $F(x) = f(x)(x-2)$

显然  $F(1) = 0, F(2) = 0$ , 又  $F(x)$  在  $[1, 2]$  上连续,  $(1, 2)$  可导,

由罗尔定理知  $\exists \xi \in (1, 2)$ , 使得  $F'(\xi) = 0$

又因为  $F'(x) = \int_1^x e^{t^2} dt + (x-2)e^{x^2} - f(x) \cdot 1 = (x-2)e^{x^2} - f(x)$

所以  $f(\xi) = (2 - \xi)e^{\xi^2}$

令  $g(x) = \frac{f(x)}{x-2}$  由罗尔定理得  $\exists \eta \in (1, 2)$

$$\frac{f(2) - f(1)}{2 - 1} = \frac{f(2)}{1} = \frac{1}{\eta} = \eta e^{\eta^2}$$

即  $f(2) = \ln 2 \cdot \eta e^{\eta^2}$

21. (本题满分 11 分)

设曲线  $y = f(x)$  可导, 且  $f'(x) > 0 (x \geq 0)$ ,  $f(x) > 0 (x \geq 0)$

曲线上任意一点  $M$  的切线与  $X$  轴交于  $T$ ,  $MP \perp x$  轴, 曲线  $y = f(x), MP, x$  轴围成的面积与  $\Delta MTP$  面积比为 3: 2, 求曲线方程.

解析: 设切点  $M$  坐标为  $(x, y)$ , 则过  $M$  的切线方程为

$$Y - y = y'(X - x)$$

$$\text{令 } Y = 0 \text{ 得 } X = x - \frac{y}{y'}$$

由题意得

$$\frac{\int_0^x f(t)dt}{\frac{1}{2} \cdot \frac{y}{y'} \cdot y} = \frac{3}{2}$$

整理并求导得  $3yy'' - 2y'^2 = 0$

令  $y' = p$   $y'' = p \frac{dp}{dy}$  代入上式得

$$3yp \frac{dp}{dy} - 2p^2 = 0$$

解得  $p = C_1 y^{\frac{2}{3}}$

即  $y' = C_1 y^{\frac{2}{3}}$

$$\frac{dy}{y^{\frac{2}{3}}} = C_1 dx$$

$3y^{\frac{1}{3}} = C_1 x + C_2$  由  $y(0) = 0$  得  $C_2 = 0$ .

22. (满分 11 分)

设二次型  $f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 + 2ax_1x_2 + 2ax_1x_3 + 2ax_2x_3$  经可逆线性变换

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = P \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \text{ 得 } g(y_1, y_2, y_3) = y_1^2 + y_2^2 + 4y_3^2 + 2y_1y_2$$

- (1) 求  $a$  的值;
- (2) 求可逆矩阵  $P$ .

解析:

$$(1) \text{ 令 } f(x_1, x_2, x_3) \text{ 的矩阵 } B = \begin{bmatrix} 1 & a & a \\ a & 1 & a \\ a & a & 1 \end{bmatrix}$$

$$f(y_1, y_2, y_3) \text{ 的矩阵 } B = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

A 与 B 合同, 则  $r(A) = r(B)$ .

由于  $|B| = 0$ , 故  $r(B) < 3$ , 故  $|A| = 0$ .

$$|A| = \begin{vmatrix} 1 & a & a \\ a & 1 & a \\ a & a & 1 \end{vmatrix} = (2a+1)(a-1)^2 = 0$$

而

$$\text{解得 } a = -\frac{1}{2} \text{ 或 } a = 1.$$

当  $a = 1$  时,  $r(A) = 1$ , 而  $r(B) = 2$ . 故

$$\text{所以 } a = -\frac{1}{2}.$$

(2) 当  $a = -\frac{1}{2}$  时, 利用配方法把  $f(x_1, x_2, x_3)$  化为规范形.

$$\begin{aligned} f(x_1, x_2, x_3) &= x_1^2 + x_2^2 + x_3^2 + x_1x_2 - x_1x_3 - x_2x_3 \\ &= \left(x_1 + \frac{1}{2}x_2 - \frac{1}{2}x_3\right)^2 - \frac{3}{4}x_2^2 + \frac{3}{4}x_3^2 - \frac{3}{2}x_2x_3 \\ &= \left(x_1 + \frac{1}{2}x_2 - \frac{1}{2}x_3\right)^2 - \frac{3}{4}(x_2 - x_3)^2 \end{aligned}$$

$$\begin{cases} z_1 = x_1 + \frac{1}{2}x_2 - \frac{1}{2}x_3 \\ z_2 = \frac{\sqrt{3}}{2}(x_2 - x_3) \\ z_3 = x_3 \end{cases} \quad \text{即令 } P_1 = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \\ 0 & 0 & 1 \end{bmatrix}$$

$Z = P_1X$ , 则  $f(x_1, x_2, x_3) = z_1^2 - z_2^2$ .

利用配方法把  $f(y_1, y_2, y_3)$  化为规范形.

$$f(y_1, y_2, y_3) = y_1^2 + y_2^2 + y_3^2 + y_1y_2 + y_3^2 = (y_1 + y_2)^2 + 4y_3^2$$



$$\text{令 } \begin{cases} z_1 = y_1 + y_2 \\ z_2 = 2y_3 \\ z_3 = y_2 \end{cases} \quad \text{即令 } P_2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix} \quad Z = P_2 Y.$$

则  $f(y_1, y_2, y_3) = z_1^2 + z_2^2$ .

故  $P_1 X = P_2 Y$  即  $X = P_1^{-1} P_2 Y$ .

所以  $P = P_1^{-1} P_2$ .

$$P_1^{-1} = \begin{bmatrix} 1 & \frac{1}{\sqrt{3}} & 1 \\ 0 & \frac{2}{\sqrt{3}} & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad P = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix}$$

由于

$$P^{-1} = \begin{bmatrix} 2 & \frac{2}{3} \\ 0 & \frac{4}{3}\sqrt{3} \\ 0 & 0 \end{bmatrix}$$

23. (本题满分 11 分)

设  $A$  为 2 阶矩阵,  $P = (\alpha, A\alpha)$ , 其中  $\alpha$  是非零向量且为  $A$  的特异向量.

(1) 证明  $P$  为可逆矩阵.

(2) 若  $A^2\alpha + A\alpha - 6\alpha = 0$ , 求  $P^{-1}AP$ , 并判断是否相似于对角矩阵.

解析:

(1)  $\alpha \neq 0$  且  $A\alpha \neq \lambda\alpha$ .

故  $\alpha$  与  $A\alpha$  线性无关.

则  $r(\alpha, A\alpha) = 2$

则  $P$  可逆.

(2) 法一: 由已知有  $A^2\alpha = -A\alpha + 6\alpha$

于是  $AP = A(\alpha, A\alpha) = (A\alpha, A^2\alpha) = (A\alpha, -A\alpha + 6\alpha)$

$= (\alpha, A\alpha) \begin{pmatrix} 0 & 6 \\ 1 & -1 \end{pmatrix}$ , 故有  $P^{-1}AP = \begin{pmatrix} 0 & 6 \\ 1 & -1 \end{pmatrix}$ ,  $\therefore P$  可逆

$\therefore$  可得  $A$  与  $\begin{pmatrix} 0 & 6 \\ 1 & -1 \end{pmatrix}$  相似, 又  $\begin{vmatrix} \lambda & -6 \\ -1 & \lambda+1 \end{vmatrix} = (\lambda+3) \cdot (\lambda-2) = 0$

$\Rightarrow \lambda_1 = -3, \lambda_2 = 2$

$\therefore$  可得  $A$  的特征值也为  $-3, 2$  于是  $A$  可相似对角化

方法二  $P^{-1}AP$  同方法一

由  $A^2\alpha + A\alpha - 6\alpha = 0$

下面是证明  $A$  可相似对角化

$(A^2 + A - 6E)\alpha = 0$

设  $(A+3E)(A-2E)\alpha = 0$

由  $(A+3E) \neq 0$  得  $(A-2E)x = 0$  有非零解

故  $(A-3E)(A-2E) = 0$

得  $(A-3E) = 0$  或  $(A-2E) = 0$

若  $(A+3E) \neq 0$  则有  $(A-2E)\alpha = 0$  故  $A\alpha = 2\alpha$  与题条件矛盾

故  $(A+3E) = 0$  同理可得  $(A-2E) = 0$

于是  $A$  的特征值为  $\lambda_1 = -3, \lambda_2 = 2$ .

$A$  有 2 个不同特征值故  $A\alpha$  相似对角化